$\sin z = 0 \Rightarrow \dfrac{e^{iz} - e^{-iz}}{2i} = 0 \Rightarrow e^{2iz} - 1 = 0 \Rightarrow e^{2iz} = 1$

$\Rightarrow 2iz = i(k2\pi) \Rightarrow z = k\pi$

$\cos z = 0 \Rightarrow \dfrac{e^{iz} + e^{-iz}}{2} = 0 \Rightarrow e^{2iz} + 1 = 0 \Rightarrow e^{2iz} = -1$

$\Rightarrow 2iz = i(\pi + k2\pi) \Rightarrow z = \frac{\pi}{2} + k\pi$

$\sinh z = 0 \Rightarrow \dfrac{e^{z} - e^{-z}}{2} = 0 \Rightarrow e^{2z} - 1 = 0 \Rightarrow e^{2z} = 1$

$\Rightarrow 2z = i(2k\pi) \Rightarrow z = k\pi i$

$\tan z = 1 \Rightarrow \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = 1 \Rightarrow e^{iz} - e^{-iz} = i(e^{iz} + e^{-iz})$

$\Rightarrow e^{2iz} - 1 = i e^{2iz} + i$

$\Rightarrow e^{2iz} = \frac{1+i}{1-i} = \sqrt{2}\, i \Rightarrow Ln(e^{2iz}) = L(\sqrt{2}\, i)$

$\Rightarrow 2iz = \ln\sqrt{2} + i(\cancel{k}\frac{\pi}{2} + k2\pi)$

$\Rightarrow z = \dfrac{i \ln 2}{4} + \left(\dfrac{\pi}{4} + k\pi\right)$

1.7 Regel van de l'Hospital toepassen bij $\frac{0}{0}$

$\lim_{z\to 0} \dfrac{\bar{z}}{z} = \lim_{(x,y)\to(0,0)} \dfrac{x - iy}{x + iy}$

langs x-as $(y=0)$: $\lim_{x\to 0} \frac{x}{x} = 1$

langs y-as $(x=0)$: $\lim_{y\to 0} \frac{-y}{y} = -1$

$\neq \Rightarrow$ lim bestaat niet

$\lim_{z\to 0} \dfrac{z}{|z|} = \lim_{(x,y)\to(0,0)} \dfrac{x+iy}{x^2+y^2}$

langs x-as $(y=0)$: $\lim_{x\to 0} \frac{x}{x^2} = \pm\infty$

langs y-as $(x=0)$: $\lim_{y\to 0} \frac{iy}{y^2} = \pm i\infty$

$\neq \Rightarrow$ lim bestaat niet

(1.8) $u(x,y) = x^3 - 3xy^2$

$$\frac{\partial^2}{\partial x^2}(x^3 - 3xy^2) = \frac{\partial}{\partial x}(3x^2 - 3y^2) = +6x$$
$$\frac{\partial^2}{\partial y^2}(x^3 - 3xy^2) = \frac{\partial}{\partial y}(-6xy) = -6x$$
$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Harmonisch toegevoegde : $u + iv$ analytisch $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\Rightarrow \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \Rightarrow v = \int 3x^2 - 3y^2 \, dy = 3xy - y^3 + C$$

(2.1) $\int_C z^2 dz$ langs parabool

(#1)

$$z = x + iy = t + it^2 \Rightarrow dz = (1 + 2it)dt$$
$$z^2 = t^2 + 2it^3 + t^4$$

$$\int_1^2 (t^2 + 2it^3 - t^4)(1 + 2it)\,dt = \dots = -\frac{86}{3} - 6i$$

(#2) $\int_C \bar{z}\, dz$ langs $z = t^2 + it \Rightarrow dz = (2t + i)dt$

$$\bar{z} = t^2 - it$$

kromme loopt van $(0,0)$ naar $(4,2) \Rightarrow t : 0 \rightarrow 2$

$$\int_C \bar{z}\, dz = \int_0^2 (t^2 - it)(2t + i)\,dt = \dots = 10 - \frac{8}{3}i$$

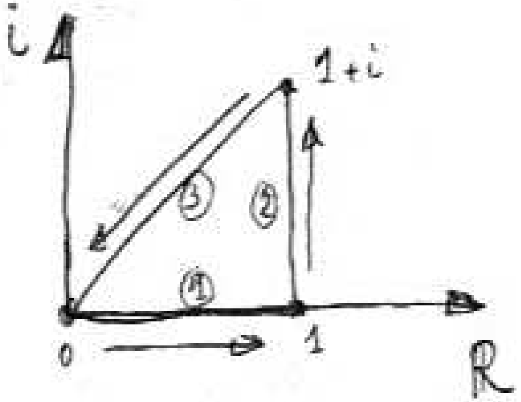
$\int_C (2z+3)dz$ met C lijnstuk $[1-2i, 3+i]$

(#3)

$$\Rightarrow z = ((3+i) - (1-2i))t + (1-2i) = (2+3i)t + 1 - 2i$$
$$\Rightarrow dz = (2+3i)dt$$

$$\Rightarrow \int_C (2z+3)dz = \int_0^1 \left[2\left((2+3i)t + 1 - 2i\right) + 3\right]\left[2 + 3i\right]dt = \dots = 17 + 19i$$

#2



$$\oint_{C^+} = -\left[\int_{①} + \int_{②} + \int_{③}\right]$$

$$= -\left[\frac{1}{2} + \frac{1}{2}i + i - \frac{3}{2} + \frac{1}{2} - \frac{3}{2}i\right] = \frac{1}{2}$$

$\int_{①}$ $[0,0] \to [1,0] \Rightarrow z = t \quad dz = dt \quad x = t \quad y = 0 \quad t: 0 \to 1$

$$\int_{①} = \int_0^1 (t + it)\,dt = (1+i)\left[\frac{t^2}{2}\right]_0^1 = \boxed{\frac{1}{2} + \frac{1}{2}i}$$

$\int_{②}$ $[1,0] \to [1,1] \Rightarrow z = it + 1 \Rightarrow dz = i\,dt \quad x = 1 \quad y = t \quad t: 0 \to 1$

$$\int_{②} = \int_0^1 (1 + i(1+t))\,i\,dt = \dots = \boxed{i - \frac{3}{2}}$$

$\int_{③}$ $[1,1] \to [0,0] \Rightarrow z = -(1+i)t + (1+i) \quad dz = -(1+i)dt \quad x = y = 1-t$

$$\int_{③} = -\int_0^1 (1-t + 2i(1-t))(1+i)\,dt = \dots = \boxed{\frac{1}{2} - \frac{3}{2}i}$$

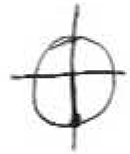
(2.2) voorwaarden: f analytisch over gebied G dat C bevat (begrensd door C
punt ligt binnen C →

$\Rightarrow$ CAUCHY

(#1) $\oint_{C^+} \frac{\cos z}{z-\pi} dz$ $\quad$ C gesloten kromme rond π → π inwendig
cos is analytisch

$$\Rightarrow \cos \pi = \frac{1}{2\pi i} \oint_{C^+} \frac{\cos z}{z-\pi} dz \Rightarrow \oint_{C^+} \frac{\cos z}{z-\pi} dz = -2\pi i$$

(#2) $\oint_{C^+} \frac{\sin 3z}{z+\frac{\pi}{2}}$ $\quad$ C is cirkel $|z| = 5$ $\quad$ $|-\frac{\pi}{2}| < 5 \Rightarrow -\frac{\pi}{2}$ inwendig
sin is analytisch



$$\Rightarrow \oint_{C^+} \frac{\sin 3z}{z+\frac{\pi}{2}} = -2\pi i \sin 3\frac{\pi}{2} = +2\pi i$$

(#5) $\oint_{C^+} \frac{z^2\, dz}{(z+2)(z-1)} = \oint_{C^+} z^2\left(\frac{1}{(z+2)(z-1)}\right) dz$

PARTIEELBREUKEN

$$\frac{A}{z+2} + \frac{B}{z-1} = \frac{A(z-1) + B(z+2)}{(z+2)(z-1)}$$

$$= \oint_{C^+} \frac{-z^2 dz}{3(z+2)} + \oint_{C^+} \frac{z^2 dz}{3(z-1)}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ -A + 2B = 1 \end{cases} \Rightarrow \begin{cases} A = -1/3 \\ B = 1/3 \end{cases}$$

* z^2 is analytisch $\quad$ $|-2| = \sqrt{5} < 3 \Rightarrow -2$ inwendig
$|1+i| = \sqrt{2} < 3 \Rightarrow 1$ inwendig

$$\Rightarrow -\frac{1}{3}\left(2\pi i (-2)^2\right) + \frac{1}{3}\left(2\pi i (1)^2\right) = -\frac{8}{3}\pi i + \frac{2}{3}\pi i = -2\pi i$$

(#2)(2.3) $e^z + z$ analytisch? $\quad e^z + z = e^{a+bi} + a + bi = e^a(\cos b + i \sin b) + a + bi$

$$= (e^a \cos b + a) + i(e^a \sin b + b) = u + iv$$

$|1-1| = 0 < 1 \Rightarrow 1$ inwendig

$$\frac{\partial u}{\partial a} = e^a \cos b \quad \frac{\partial v}{\partial b} = e^a \cos b \quad \frac{\partial u}{\partial a} = \frac{\partial v}{\partial b} \Rightarrow \text{anal.}$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{C^+} \frac{f(z)\, dz}{(z-a)^{n+1}} \qquad \text{hier: } n = 3 \quad f^{(3)}a = e^a = e$$

$$\Rightarrow \oint_{C^+} \frac{e^z + z}{(z-1)^4} dz = \frac{2\pi i\, f^{(n)}a}{n!} = \frac{\pi i e}{3}$$

Cauchy-integraal

> Indien f analytisch is over een enkelvoudig samenhangend gebied G dat begrensd is door de kromme C en indien a een inwendig punt is van G, dan geldt:
>
> $$f(a) = \frac{1}{2\pi i}\int_{C^+}\frac{f(z)}{z-a}dz \qquad \text{en} \qquad f^{(n)}(a) = \frac{n!}{2\pi i}\int_{C^+}\frac{f(z)}{(z-a)^{n+1}}dz$$
>
> waarbij C^+ de omloopszin is die G links laat liggen.

In de volgende oefeningen kunnen de integralen op een eenvoudige manier berekend worden met de formule van Cauchy. Waarom kon dit tot nu toe niet?

Oefening 2.2 Bereken volgende integralen m.b.v. Cauchy:

#1 $\oint_{C^+}\frac{\cos z}{z-\pi}dz$, waarbij C een gesloten kromme rond π is.

#2 $\oint_{C^+}\frac{\sin 3z}{z+\frac{\pi}{2}}dz$, waarbij C de cirkel $|z|=5$ is.

#3 $\oint_{C^+}\frac{\cos \pi z}{z-1}dz$, waarbij C een gesloten kromme rond 1 is.

#4 $\oint_{C^+}\frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz$, waarbij C de cirkel $|z|=3$ is.

#5 $\oint_{C^+}\frac{z^2}{(z+2)(z-1)}dz$, waarbij C de cirkel $|z+i|=3$ is.

#6 $\oint_{C^+}\frac{dz}{z(z^2-1)}$, waarbij C de cirkel $|z-2|=5/2$ is.

Oefening 2.3 Bereken volgende integralen m.b.v. Cauchy:

#1 $\oint_{C^+}\frac{e^{2z}}{(z+1)^4}dz$, waarbij C de cirkel $|z|=3$ is.

#2 $\oint_{C^+}\frac{e^z+z}{(z-1)^4}dz$, waarbij C de cirkel $|z-1|=1$ is.

#3 $\oint_{C^+}\frac{ze^z}{(z-2)^3}dz$, waarbij C de cirkel $|z-2|=1$ is.

#4 $\oint_{C^+}\frac{\cos(\pi z)(3z+1)}{z^3-3z+2}dz$, waarbij C de cirkel $|z-1|=1$ is.

#5 $\oint_{C^+}\frac{z}{(z-3)^2(2z+1)}dz$, waarbij C de cirkel $|z|=4$ is.

#6 $\oint_{C^+}\frac{2z^2+17z+15}{(z+2)^2(z+1)z}dz$, waarbij C de cirkel $|z+2|=3$ is.

2.3 #5 $\oint_{C^+}\frac{z\,dz}{(z-3)^2(2z+1)}$ *

$$\frac{1}{(z-3)^2(2z+1)} = \frac{A}{(z-3)} + \frac{B}{(z-3)^2} + \frac{C}{(z+\frac{1}{2})}$$

$\Rightarrow A = -\frac{2}{49} \quad B = \frac{1}{7} \quad C = \frac{8}{49}$

$\Rightarrow * = \oint_{C^+} -\frac{2z\,dz}{49(z-3)} + \oint_{C^+}\frac{z\,dz}{7(z-3)^2} + \oint_{C^+}\frac{2z\,dz}{49(z+\frac{1}{2})}$

Z is analytisch
alles binnen de cirkel

$\Rightarrow * = -\frac{2}{49}2\pi i(3) + \frac{2\pi i}{7} - \frac{2\cdot 2\pi i}{49\cdot 2}$

$= 0$

3 mogelijkheden : analytisch + inwendig $\Rightarrow$ Cauchy

analytisch + niet inwendig $\Rightarrow$ Cauchy-Goursat $\Rightarrow \int = 0$

niet analytisch $\Rightarrow$ GEWOON UITREKENEN = FUCK

2.4 #1 1) e^z analytisch $\quad |2| = 2 < 3 \Rightarrow$ inwendig $\Rightarrow$ CAUCHY

$$\oint_{C^+} \frac{e^z}{z-2} dz = 2\pi i e^2$$

#1 2) e^z analytisch $\quad |2| = 2 > 1 \Rightarrow$ niet inwendig $\Rightarrow$ CAUCHY-GOURSAT

$$\oint_{C^+} \frac{e^z}{z-2} = 0$$

#3 1) $\oint_{C^+} \frac{z\,dz}{2z-5} = \frac{1}{2} \oint_{C^+} \frac{z\,dz}{z-\frac{5}{2}}$ $\qquad$ z analytisch

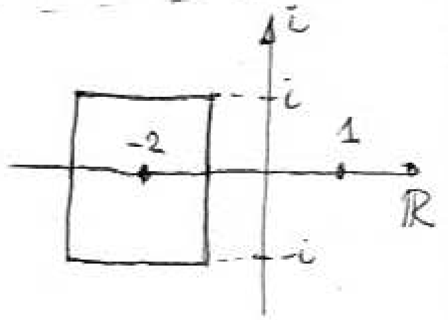
$|\frac{5}{2}| = \frac{5}{2} > 2 \Rightarrow$ niet inwendig

$\Rightarrow$ CAUCHY-GOURSAT : $\oint_{C^+} = 0$

#3 2) z analytisch $\quad |\frac{5}{2} - 3| = \frac{1}{2} < 2 \Rightarrow$ inwendig $\rightarrow$ CAUCHY-GOURSAT

$$\oint_{C^+} \frac{z\,dz}{2z-5} = \frac{1}{2} \oint_{C^+} \frac{z\,dz}{z-\frac{5}{2}} = \frac{1}{2} 2\pi i \frac{5}{2} = \frac{5\pi i}{2}$$

#6 1) $$\oint_{C^+} \frac{z^2}{(z+2)(z-1)} \overset{\text{ZIE 2.2 \#5}}{=} -\frac{1}{3} \oint_{C^+} \frac{z^2 dz}{(z+2)} + \frac{1}{3} \oint_{C^+} \frac{z^2}{(z-1)}$$



-2 ligt binnen rechthoek, +1 niet

z^2 is analytisch

$$\Rightarrow \oint_{C^+} = -\frac{1}{3} 2\pi i (-2)^2 = -\frac{8}{3}\pi i$$

#6 2) $$\oint_{C^+} = -\frac{1}{3} \oint_{C^+} \frac{z^2 dz}{z+2} + \frac{1}{3} \oint_{C^+} \frac{z^2}{z-1}$$ $\qquad$ z^2 analytisch

$|-2| = 2 > \sqrt{2} \Rightarrow$ niet inwendig $\quad |1| = 1 < \sqrt{2} \Rightarrow$ inwendig

$$\rightarrow \oint_{C^+} = \frac{1}{3} 2\pi i (-1)^2 = \frac{2}{3}\pi i$$

3.1 #1 $R = \lim_{n\to\infty} \left| \dfrac{\frac{z^n}{\sqrt{n+1}}}{\frac{z^{n+1}}{\sqrt{n+2}}} \right| = \left| \dfrac{1}{z} \dfrac{n\sqrt{1+\frac{2}{n}}}{n\sqrt{1+\frac{1}{n}}} \right| = 1$

op de rand : $z = e^{i\theta}$

$\Rightarrow \sum_{n=0}^{\infty} \frac{e^{in\theta}}{\sqrt{n+1}} = *$ $\quad w_n = \frac{1}{\sqrt{n+1}} \xrightarrow{n\to\infty} 0$

$w_n \geqslant w_{n+1}$ niet stijgend OK WANT $\sqrt{n+1} < \sqrt{n+2}$

$\Rightarrow *$ convergent voor $\theta \neq 2k\pi$

$\theta = 2k\pi \Rightarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \geqslant \sum_{n=0}^{\infty} \frac{1}{n+1} \Rightarrow$ divergent want $\sum \frac{1}{n+1}$ div

$\Rightarrow$ convergentiegebied : $\{ z \mid |z| \leq 1 \} / \{1\}$

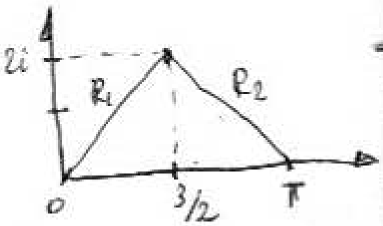
#3 $R = \lim_{n\to\infty} \left| \dfrac{\frac{z^n}{n}}{\frac{z^{n+1}}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1}{z} \left(\frac{n+1}{n} \right) \right| = 1$

op de rand : $z = e^{i\theta} \Rightarrow \sum_{n=0}^{\infty} \frac{e^{in\theta}}{n}$ $\quad w_n = \frac{1}{n} \xrightarrow{n\to\infty} 0$ $\Rightarrow$ CONV

$w_n \geqslant w_{n+1}$ ok $\frac{1}{n} \geqslant \frac{1}{n+1}$

$\theta = 2k\pi \quad \sum_{n=0}^{\infty} \frac{1}{n}$ is divergent $\Rightarrow$ gebied : $\{ |z| \leq 1 \} / \{1\}$

3.2 #1 $\csc z = \frac{1}{\sin z}$ $\Rightarrow$ singulier als $\sin z = 0 \Rightarrow z = k\pi$



$R_1 = \sqrt{\frac{9}{4} + 4} = \frac{5}{2} \quad R_2 = \sqrt{\left(\frac{3}{2} - \pi\right)^2 + 4}$

$= \sqrt{\frac{9}{4} - 3\pi + \pi^2 + 4}$. $\quad \pi^2 > 3\pi \Rightarrow R_2 > R_1$

$\Rightarrow$ convergentiestraal $= R_1 = \frac{5}{2}$

#3 $\frac{1}{1+e^{7z}}$ singulier al $e^{7z} = -1 \Rightarrow 7z = i(\pi + k2\pi) \Rightarrow z = i\left(\frac{\pi}{7} + k\frac{2}{7}\pi\right)$

$\frac{\pi}{7} i$ en $\frac{3\pi}{7} i$ liggen het dichtst bij i

$\left| \frac{\pi}{7} - 1 \right| = \frac{|\pi - 7|}{7} \approx \frac{4}{7} \quad \left| \frac{3\pi}{7} - 1 \right| = \frac{|3\pi - 7|}{7} \approx \frac{9-7}{7} \approx \frac{2}{7} \Rightarrow R = |3\pi - 7| \frac{1}{7}$

3.1 #5 $\sum_{n=1}^{\infty} \frac{1}{n^2 3^n}\left|\frac{z+1}{z-1}\right|^n$ $\quad R = \lim_{n\to\infty}\left|\frac{(n+1)^2 3^{\cancel{n+1}}}{n^2 \cancel{3^n}}\right| = 3$

op de rand: $\left|\frac{z+1}{z-1}\right| = Re^{i\theta}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\cancel{3^n} e^{in\theta}}{n^2 \cancel{3^n}} = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n^2}$ $\quad w_n = \frac{1}{n^2} \xrightarrow{n\to\infty} 0$

$w_n \geq w_{n+1}$ $\quad \frac{1}{n^2} \geq \frac{1}{(n+1)^2}$ ok

$\Rightarrow$ convergent op de rand zonder 1

$\theta = k2\pi \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent $\rightarrow$ gebied $= \left\{\left|\frac{z+1}{z-1}\right| \leq 3\right\}$

3.1 #6 $\sum_{n=1}^{\infty} \frac{3^n + 2^n}{n 6^n}(z-3)^n$ * $\quad R = \lim_{n\to\infty}\left|\frac{\frac{3^n+2^n}{n6^n}}{\frac{3^{n+1}+2^{n+1}}{(n+1)6^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{3^n+2^n}{3^{n+1}+2^{n+1}}\left(\frac{n+1}{n}\right)6\right|$

$= \lim_{n\to\infty}\left|\frac{3^n+2^n}{3\cdot3^n+2\cdot2^n}\left[\frac{n+1}{n}\right]6\right| = \lim_{n\to\infty}\left|\frac{3^n+2^n}{\cancel{6}\left(\frac{3^n}{2}+\frac{2^n}{3}\right)}\left(\frac{n+1}{n}\right)\cancel{6}\right| = \lim_{n\to\infty}\left|\frac{\cancel{3^n}\left(1+\left(\frac{2}{3}\right)^n\right)}{\cancel{3^n}\left(\frac{1}{2}+\frac{1}{3}\left(\frac{2}{3}\right)^n\right)}\right|\frac{n+1}{n}$

$= \lim_{n\to\infty} a^n$ met $a<1$ $= 0 \Rightarrow \lim_{n\to\infty}\left|\frac{1+\left(\frac{2}{3}\right)^n}{\frac{1}{2}+\frac{1}{3}\left(\frac{2}{3}\right)^n}\left[\frac{n+1}{n}\right]\right| = 2$

op de rand: $(z-3) = 2e^{i\theta} \Rightarrow$ * $= \sum_{n=1}^{\infty} \frac{3^n+2^n}{n6^n}\cdot 2^n e^{in\theta}$

$w_n = \frac{6^n+4^n}{n6^n}$ $\quad \lim_{n\to\infty} w_n = \lim_{n\to\infty}\frac{6^n+4^n}{n6^n} = \frac{\cancel{6^n}\left(1+\left(\frac{4}{6}\right)^n\right)}{n\cancel{6^n}} = 0$

$w_n \geq w_{n+1}$? $\quad \frac{1+\left(\frac{4}{6}\right)^n}{n} \overset{?}{\geq} \frac{1+\left(\frac{4}{6}\right)^{n+1}}{n+1} \Leftrightarrow (n+1)\left(1+\left(\frac{2}{3}\right)^n\right) - n\left(1+\left(\frac{2}{3}\right)^{n+1}\right) \overset{?}{\geq} 0$

$\Leftrightarrow 1 + n\left(\frac{2}{3}\right)^n + \left(\frac{2}{3}\right)^n - n\left(\frac{2}{3}\right)^{n+1} \geq 0 \Leftrightarrow 1 + \left(\frac{2}{3}\right)^n\left(1+\frac{1}{3}n\right) \geq 0$ ✓

gebied $= \{|z-3| \leq 2\} / \{5\}$

$\theta = 2k\pi \Rightarrow z-3 = 2 \Rightarrow \sum \frac{1+\left(\frac{2}{3}\right)^n}{n} \geq \sum \frac{1}{n} \Rightarrow$ div

3.3 #1 $R_1 = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = 1 \qquad R_2 = \lim_{n\to+\infty} \left|\frac{-n}{-n-1}\right| = 1$

op de rand: $z = e^{i\theta} \Rightarrow \sum \frac{e^{in\theta}}{n} \qquad \omega_n = \frac{1}{n} \xrightarrow{n\to\infty} 0$

$\omega_n > \omega_{n+1} \quad \frac{1}{n} > \frac{1}{n+1}$

$\theta = 2k\pi \Rightarrow \sum \frac{1}{n}$ divergent $\Rightarrow$ gebied is $\{|z| = 1\} \setminus \{1\}$

#5 $R_1 = \lim_{n\to\infty} \left|\frac{(n+1)\,3^{|n+1|}}{n\,3^{|n|}}\right| = \left|\frac{n+1}{n}\cdot 3\right| = 3$

$R_2 = \lim_{n\to\infty} \left|\frac{-n\,3^{|n|}}{(-n-1)\,3^{|-n-1|}}\right| = \left|-\frac{n}{n+1}\cdot\frac{1}{3}\right| = \frac{1}{3}$

RAND 1: $z = 3e^{i\theta} \Rightarrow \sum_{n=0}^{\infty} \frac{3^n e^{in\theta}}{n\,3^{|n|}}$ convergent (zie vorige opg) divergent in 3

RAND 2: $z = \frac{1}{3}e^{i\theta} \Rightarrow \sum_{n=-1}^{-\infty} \frac{e^{in\theta}}{n\,3^{|n|}\,3^n} = \sum_{n=-1}^{-\infty} \frac{e^{in\theta}}{n}$ convergent behalve in $\frac{1}{3}$

$\Rightarrow$ GEBIED IS $\{\frac{1}{3} \leq |z| \leq 3\} \setminus \{3, 1/3\}$

3.4 #1 $f(z) = \frac{e^{2z}}{(z-1)^3}$ $*$ $\qquad e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!} \Rightarrow e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!}$

$\Rightarrow$ ~~$* = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!\,(z-1)^3}$~~ $\quad * = \frac{e^{2(z-1+1)}}{(z-1)^3} = \frac{e^2 e^{2(z-1)}}{(z-1)^3} = \frac{e^2}{(z-1)^3} \sum_{n=0}^{\infty} \frac{2^n (z-1)^n}{n!}$

$= e^2 \sum_{n=0}^{\infty} \frac{2^n (z-1)^{n-3}}{n!}$ INDEX OPSCHUIVEN $k = n-3 \Rightarrow e^2 \sum_{k=-3}^{\infty} \frac{2^{k+3}(z-1)^k}{(k+3)!}$

RESIDU: $k=-1$ EN INVULLEN

Oefening 3.3 Bepaal het convergentiegebied van volgende Laurentreeksen:

#1 $\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{z^n}{n}$ #2 $\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{\log n}{n^2} z^n$ #3 $\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{z^n}{n^2}$

#4 $\sum_{n=-\infty}^{+\infty} \frac{z^n}{|n|!}$ #5 $\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{z^n}{n3^{|n|}}$ #6 $\sum_{n=-\infty}^{+\infty} \frac{z^n}{3^{n^2}}$

Indien f analytisch is in $R_2 < |z-a| < R_1$, dan kan f geschreven worden als een Laurentreeks

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-a)^n \qquad \text{voor } R_2 < |z-a| < R_1$$

De coëfficiënt van $(z-a)^{-1}$ noemt men het **residu** van f in a

$$a_{-1} = \operatorname{Res}(f,a)$$

Oefening 3.4 Bepaal de Laurentreeks van de gegeven functie f in een omgeving van de gegeven singulariteit. Bepaal ook het residu.

#1 $f(z) = \frac{e^{2z}}{(z-1)^3}$ rond $z=1$

#2 $f(z) = \frac{z}{(z+1)(z+2)}$ rond $z=-2$

#3 $f(z) = e^{\frac{z}{z-2}}$ rond $z=2$

#4 $f(z) = (z-3)\sin\frac{1}{z+2}$ rond $z=-2$

#5 $f(z) = \frac{z-\sin z}{z^3}$ rond $z=0$

#6 $f(z) = \frac{1}{z^2(z-3)^2}$ rond $z=3$

3.4 #3 $e^{\frac{z}{z-2}} = e^{\frac{z-2+2}{z-2}} = e^{\left(1+\frac{2}{z-2}\right)}$

$= e\, e^{\frac{2}{z-2}} = e\sum_{n=0}^{\infty} \frac{2^n (z-2)^{-n}}{n!}$

RESIDU $= 2e$

3.4 #6 TRUKJE: $\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} \quad |z|<1$

$f(z) = \frac{1}{(z-3)^2}\cdot\frac{1}{z^2} = \frac{1}{(z-3)^2}\cdot\frac{1}{(z-3+3)^2}$

$= \frac{1}{(z-3)^2}\cdot\frac{1}{9\left(\frac{z-3}{3}+1\right)^2} = \frac{1}{9(z-3)^2}\sum_{n=1}^{\infty} n\left(\frac{3-z}{3}\right)^{n-1}$

$= \frac{1}{9}\sum_{n=1}^{\infty} n\frac{(3-z)^{n-3}}{(3)^{n-1}} = \frac{1}{9}\sum_{n=1}^{\infty} n\left(-\frac{1}{3}\right)^{n-1}(z-3)^{n-3}$

$= \frac{1}{9}\sum_{k=-2}^{\infty} (k+3)\left(-\frac{1}{3}\right)^{k+2}(z-3)^k$

RESIDU $= \frac{1}{9}(2)\left(-\frac{1}{3}\right) = -\frac{2}{27}$

#4 $\sum \frac{z^n}{|n|!}$ $R_1 = \lim_{n\to\infty} \frac{|n+1|!}{|n|!} = \frac{(n+1)\cancel{n!}}{\cancel{n!}} = \lim_{n\to\infty} n+1 = \infty$

$R_2 = \lim_{n\to\infty} \frac{|-n|!}{|-n-1|!} = \frac{n!}{(n+1)!} = \frac{1}{n+1} = 0$

OP DE RAND: $z = 0 \Rightarrow \sum \frac{0}{|n|!}$???

#6 $\sum \frac{z^n}{3^{n^2}}$ $R_1 = \frac{3^{(n+1)^2}}{3^{n^2}} = \frac{3^{\cancel{n^2}+2n+1}}{\cancel{3^{n^2}}} = 3^{2n+1} = \infty$

$R_2 = \frac{3^{(-n)^2}}{3^{(-n-1)^2}} = \frac{\cancel{3^{n^2}}}{3^{\cancel{n^2}+2n+1}} = 0$

RAND $z = 0$???

(4.1) #1 $f(z) = e^{-\frac{1}{z}}$ singulariteit voor $z = 0$

orde: $\lim\limits_{z\to 0} e^{-\frac{1}{z}} z^k < \infty \Rightarrow k = \infty$ omdat de e-macht te groot is

$\rightarrow$ ESSENTIELE SINGULARITEIT

RESIDU VIA LAURENTREEKS: $e^{-\frac{1}{z}} = \sum\limits_{n=0}^{\infty} \frac{\left(-\frac{1}{z}\right)^n}{n!} = \sum\limits_{n=0}^{\infty} \frac{(-1)^{-n}(z)^{-n}}{n!} \rightarrow$ RES $= -1$

#2 $f(z) = \tan(z)$ singulier voor $z = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$

orde: $\lim\limits_{z\to\frac{\pi}{2}+k\pi} \tan z \left(z - \frac{\pi}{2}\right) = \lim\limits_{z\to\frac{\pi}{2}} \frac{\sin(z)}{\cos(z)}\left(z-\frac{\pi}{2}\right) = \frac{0}{0} \overset{H}{=} \lim\limits_{z\to\frac{\pi}{2}} \frac{\cos(z)\left(z-\frac{\pi}{2}\right)+\sin(z)}{-\sin(z)} = -1 < \infty$

$\rightarrow$ het is een pool van orde 1 $\Rightarrow \mathrm{Res}\left(f, \frac{\pi}{2}+k\pi\right) = -1$

#4 $f(z) = \frac{z}{(z-1)(z-2)^2}$ singulier voor $z = 1$ en $z = 2$

$\boxed{z=1}$ orde: $\lim\limits_{z\to 1} \frac{z\cancel{(z-1)}}{\cancel{(z-1)}(z-2)^2} = 1 \rightarrow$ pool orde 1 en residu $= 1$

$\boxed{z=2}$ orde: $\lim\limits_{z\to 2} \frac{z\cancel{(z-2)^2}}{(z-1)\cancel{(z-2)^2}} = 2 \rightarrow$ pool orde 2

$\mathrm{Res}(f,2) = \lim\limits_{z\to 2} \frac{d}{dz}\left(\frac{z}{(z-1)}\right) = \lim\limits_{z\to 2} \frac{(z-1)-z}{(z-1)^2} = -1$

#13 $f(z) = e^{\left(\frac{1}{z-1}\right)} + \frac{z}{(z^2-1)}$ singulier voor $z = 1$ en $z = -1$

$\boxed{z=-1}$ orde: $\lim\limits_{z\to -1} e^{\left(\frac{1}{z-1}\right)}(z+1) + \frac{z\cancel{(z+1)}}{(z-1)\cancel{(z+1)}} = \frac{1}{2} = \mathrm{Res}(f,-1)$

$\boxed{z=1}$ orde: $\lim\limits_{z\to 1} \left(e^{\left(\frac{1}{z-1}\right)}(z-1)^k + \frac{z(z-1)^k}{(z-1)(z+1)}\right)$ k moet oneindig groot worden om de e-macht te compenseren $\rightarrow$ ess.

LAURENTREEKS: $e^{\left(\frac{1}{z-1}\right)} = \sum\limits_{n=0}^{\infty} \frac{(z-1)^{-n}}{n!}$ $\quad \frac{z}{(z+1)(z-1)} = \frac{z+1-1}{(z+1)(z-1)} = \frac{-1}{(z-1)(z-1+2)} \overset{+\frac{1}{z-1}}{=} \frac{-1}{2(z-1)\left(\frac{z-1}{2}\ldots\right.}$

$= \left(-\frac{1}{2}\right)\left(\frac{1}{z-1}\right)\left(\frac{1}{\left(1-\left(-\frac{1}{2}\right)(z-1)\right)}\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{z-1}\right)\sum\limits_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(z-1)^n = \sum\limits_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1}(z-1)^{n-1}$

$\rightarrow$ LAURENTREEKS $= \sum\limits_{n=0}^{\infty} \frac{(z-1)^{-n}}{n!} + \frac{1}{z-1} + \sum\limits_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1}(z-1)^n \rightarrow$ RES $= \frac{3}{2}$

4.2 #3 $\oint_{C^+} \frac{e^{zt}}{z^2+1}\,dz = 2\pi i \sum \mathrm{Res}(f,\alpha)$ $\quad f = \frac{e^{zt}}{z^2+1} = \frac{e^{zt}}{(z+i)(z-i)}$

f is singulier in $\pm i$ en beide singulariteiten liggen binnen C

$\boxed{+i}$ orde 1 $\Rightarrow \mathrm{Res}(f,i) = \lim_{z\to i} \frac{e^{zt}(z-i)}{(z+i)(z-i)} = \frac{e^{it}}{2i}$

$\boxed{-i}$ orde 1 $\Rightarrow \mathrm{Res}(f,-i) = \lim_{z\to -i} \frac{e^{zt}\cancel{(z+i)}}{\cancel{(z+i)}(z-i)} = \frac{e^{-it}}{-2i}$

$\Rightarrow \oint_{C^+} = 2\pi i\left(\frac{e^{it}-e^{-it}}{2i}\right) = 2\pi i \sin t$

#2 $\oint_{C^+} \frac{z^2\,dz}{(z+1)(z+2)}$ singulier in -1 en -2 beide liggen binnen C

$\boxed{-1}$ orde 1 $\Rightarrow \mathrm{Res}(f,-1) = \lim_{z\to -1} \frac{z^2\cancel{(z+1)}}{\cancel{(z+1)}(z+2)} = 1$

$\boxed{-2}$ orde 1 $\Rightarrow \mathrm{Res}(f,-2) = \lim_{z\to -2} \frac{z^2\cancel{(z+2)}}{(z+1)\cancel{(z+2)}} = -4$

$\oint_{C^+} = 2\pi i(1-4) = -6\pi i$

#5 $\oint_{C^+} z e^{1/z}\,dz$ singulier in 0 ligt binnen C

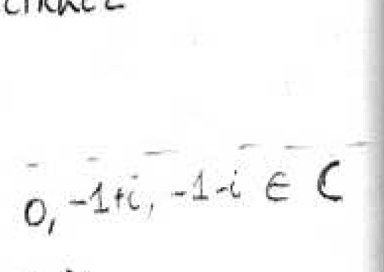
$\boxed{0}$ orde ∞ essentiële singulariteit (e-macht kan niet gecompenseerd worden)

RES VIA LAURENT: $z e^{1/z} = z\sum_{n=0}^{\infty} \frac{(z)^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{(z)^{1-n}}{n!} = \sum_{k=1}^{-\infty} \frac{z^k}{(-k+1)!}$ $\quad 1-n=k \Rightarrow n=-k+1$

$\Rightarrow \mathrm{Res}(f,0) = \frac{1}{2} \Rightarrow \oint_{C^+} = \pi i$

#6 $\oint_{C^+} \frac{(1-z)e^{\sqrt{\pi}z}}{(z+24)^{23}\sin \pi z}\,dz$ singulier in $\mathbb{Z}\setminus\{1\}$

want: $\lim_{z\to 1} f = \frac{0}{0} \overset{H}{=} \frac{e^{\sqrt{\pi}}}{\pi\, 25^{23}} \neq \infty$



ER VALT GEEN ENKELE SINGULARITEIT BINNEN DE CIRKEL

f is analytisch (behalve in $\mathbb{Z}\setminus\{1\}$) $\Rightarrow \oint_{C^+} = 0$

#4 $\oint_{C^+} \frac{e^{tz}}{z^2(z^2+2z+2)} = \oint_{C^+} \frac{e^{tz}}{z^2(z+1-i)(z+1+i)}$ 3 singulariteiten $0, -1+i, -1-i \in C$

$\boxed{0}$ 2e orde pool $\Rightarrow \mathrm{Res} = \lim_{z\to 0} \frac{d}{dz}\left(\frac{e^{tz}\cancel{z^2}}{\cancel{z^2}(z+1-i)(z+1+i)}\right) = \dots = \frac{-1+t}{2}$

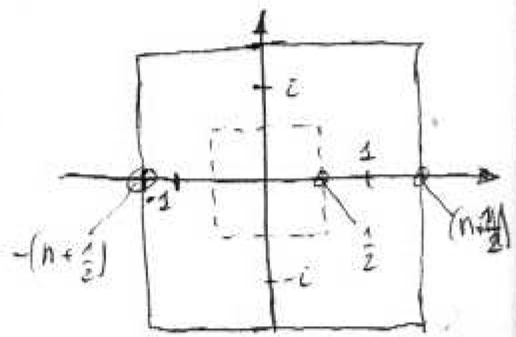
$\boxed{-1+i}$ 1e orde pool $\Rightarrow \mathrm{Res} = \lim_{z\to -1+i}\left(\frac{e^{tz}\cancel{(z+1-i)}}{z^2\cancel{(z+1-i)}(z+1+i)}\right) = \dots = \frac{e^{-t+ti}}{+4}$

$\boxed{-1-i}$ 1e orde pool $\Rightarrow \mathrm{Res} = \lim_{z\to -1-i}\left(\frac{e^{tz}\cancel{(z+1+i)}}{z^2(z+1-i)\cancel{(z+1+i)}}\right) = \dots = \frac{e^{-t-ti}}{+4}$

$\Rightarrow \oint_{C^+} = \pi i\left(t-1+e^{-t}\cos\right.$

VOORBEELD P.145 $\sum_{-\infty}^{\infty} \frac{1}{(n-\frac{1}{2})^2}$

C_n :



① $I_n = \oint_{C_n} \frac{\cot(\pi z)\, dz}{(z-1/2)^2}$

② cont? singulier in $1/2$ en $k \in \mathbb{Z}$

$\Rightarrow$ continu op C_n $(n>0)$

Begrensd? $\left|\frac{\cot(\pi z)}{(z-\frac{1}{2})^2}\right| \leq \frac{M}{|z-\frac{1}{2}|^2} * \quad \boxed{(z-\tfrac{1}{2})^2 \geq |n+\tfrac{1}{2}-\tfrac{1}{2}|^2 = n^2} \quad * \leq \frac{M}{n^2}$

$\hookrightarrow$ z binnen C_n

③ $|I_n| = \left|\oint_{C_n} \frac{\cot(\pi z)}{(z-\frac{1}{2})^2}\right| \leq \oint_{C_n} \left|\frac{\cot(\pi z)}{(z-\frac{1}{2})^2}\right| dz \leq \oint_{C_n} \frac{M}{n^2} dz = \frac{M}{n^2} \oint_{C_n} dz = \frac{M}{n^2} 4.2.(n+\frac{1}{2})$

$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \frac{M}{n^2} 4.2(n+\frac{1}{2}) = 0$

④ $I_n = 2\pi i \sum \text{Res} \Rightarrow \text{Res}(f, \frac{1}{2}) = \lim_{z\to\frac{1}{2}} \frac{d}{dz} \cot(\pi z) = \lim_{z\to\frac{1}{2}} -\frac{\pi}{\sin^2 \pi z} = -\pi$

$\text{Res}(f, k) = \lim_{z\to k} \frac{\cos \pi z}{\sin \pi z} \frac{(z-k)}{(z-\frac{1}{2})^2} = \frac{(-1)^k}{(k-\frac{1}{2})^2} \lim_{z\to k}\left(\frac{z-k}{\sin \pi z}\right) \overset{H}{=} \frac{1}{\pi(k-\frac{1}{2})^2}$

$\Rightarrow I_n = 2\pi i\left(-\pi + \sum_{k=-n}^{n} \frac{1}{\pi(k-\frac{1}{2})^2}\right)$

⑤ $\lim_{n\to\infty} |I_n| = \lim_{n\to\infty} \left(2\pi i\left(-\pi + \sum_{k=-n}^{n} \frac{1}{\pi(k-\frac{1}{2})^2}\right)\right) \Leftrightarrow 0 = -\pi + \sum_{k=-\infty}^{\infty} \frac{1}{\pi(k-\frac{1}{2})^2}$

$\Leftrightarrow \pi^2 = \sum_{-\infty}^{\infty} \frac{1}{(k-\frac{1}{2})^2}$

4.4 #1 $\sum_{-\infty}^{+\infty} \frac{(-1)^n}{(n-\frac{1}{4})^2}$ ① $I_n = \oint_{C_n} \frac{\operatorname{cosec} \pi z}{(z-\frac{1}{4})^2}$

② singulier in $\frac{1}{4}$ en $k \in \mathbb{Z}$ $\notin C_n$ $\Rightarrow$ continu op C_n

$\left|\frac{\operatorname{cosec} \pi z}{(z-\frac{1}{4})^2}\right| \leq \frac{M}{|(z-1/4)^2|}$ * $||z-1/4|^2| \leq |n+\frac{1}{2}-1/4|^2 = (n+1/4)^2 \Rightarrow * \leq \frac{M}{(n+\frac{1}{4})^2}$

③ $|I_n| \leq \oint_{C_n} \left|\frac{\operatorname{cosec} \pi z}{(z-\frac{1}{4})^2}\right| dz \leq \frac{M}{(n+\frac{1}{4})^2} \oint_{C_n} dz \leq \frac{M \cdot 4(2n+1)}{(n+\frac{1}{4})^2}$ $\lim_{n\to\infty} (I_n) = 0$

④ $I_n = 2\pi i \sum \mathrm{Res}$ $\mathrm{Res}(f, \frac{1}{4}) = \lim_{z\to\frac{1}{4}} \frac{d}{dz} \operatorname{cosec} \pi z = \lim_{z\to\frac{1}{4}} -\frac{\pi \cos \pi z}{\sin^2 \pi z} = -\pi\sqrt{2}$

$\mathrm{Res}(f, k) = \lim_{z\to k} \frac{\operatorname{cosec} \pi z\,(z-k)}{(z-\frac{1}{4})^2} = \frac{(-1)^k}{\pi(k-\frac{1}{4})^2}$

⑤ $|I_n| = 2\pi i \sum \mathrm{Res} \Rightarrow \sum_{-\infty}^{+\infty} \frac{(-1)^k}{(k-\frac{1}{4})^2} = \pi^2\sqrt{2}$

5.1 #1 $$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \oint_{C^+} \frac{-i\,dz}{z\left(5+4\,\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)} = \oint_{C^+} \frac{dz}{2z^2-5iz-2} = \oint_{C^+} \frac{dz}{2(z-2i)(z-\frac{i}{2})}$$

singulier in $2i$ en $\frac{i}{2}$. enkel $i/2$ ligt binnen de eenheidscirkel

$$\mathrm{Res}(f, i/2) = \lim_{z\to i/2} \frac{(z-i/2)}{2(z-2i)(z-i/2)} = -\frac{i}{3} \qquad \oint_{C^+} = -2\pi i \cdot \frac{i}{3} = \frac{2}{3}\pi$$

#2 $$\int_0^{2\pi} \frac{d\theta}{2-\cos\theta} = \oint_{C^+} \frac{-i\,dz}{z\left(2-\frac{z+1/z}{2}\right)} = \oint_{C^+} \frac{-2i\,dz}{4z-z^2-1} = \oint_{C^+} \frac{+2i\,dz}{z^2-4z+1}$$

singulier in $2\pm\sqrt{3}$, enkel $2-\sqrt{3}$ ligt binnen de eenheidscirkel

$$\mathrm{Res}(f, 2-\sqrt{3}) = \lim_{z\to 2-\sqrt{3}} \frac{2i}{(z-(2+\sqrt{3}))} = \frac{2i}{-2\sqrt{3}} = -\frac{i}{\sqrt{3}} \Rightarrow \oint_{C^+} = -2\pi i\,\frac{i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

(#2) (5.2)

- $Q(x)$ geen reele nulpunten ✓
- $Gr(Q) - Gr(P) \geq 2 \quad 4 - 1 = 3 \geq 2$ ✓
- P en Q geen gemeenschappelijk factor ✓

$\Bigg\} \displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2+4)(x^2+1)} = 2\pi i \sum Res$

singulier in $\pm 2i$ en $\pm i$ enkel $2i$ en i liggen in Bovenhalfvlak (BHV)

$$\left.\begin{aligned} Res(f, 2i) &= \lim_{z \to 2i} \frac{1}{(z+2i)(z^2+1)} = \frac{+i}{12} \\ Res(f, i) &= \lim_{z \to i} \frac{1}{(z^2+4)(z+1)} = -\frac{i}{6} \end{aligned}\right\} 2\pi i \sum Res = 2\pi i \left(\frac{i}{12} - \frac{i}{6}\right) = \frac{\pi}{6}$$

(#5) Voldoet aan de vwdn $\Rightarrow \displaystyle\int_{-\infty}^{+\infty} \frac{dx}{[illegible]} = 2\pi i \sum Res$

singulier in $\pm i$ enkel $+i$ ligt in BHV

$$Res(f, i) = \lim_{z \to i} \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right) = \lim_{z \to i} -\frac{2(z+i)}{(z+i)^4} = \frac{-i}{4} \Rightarrow \int_{-\infty}^{+\infty} = -\frac{2\pi i \cdot i}{4} = \frac{\pi}{2}$$

(#6) $\dfrac{1}{(x^2+4)^2}$ IS EEN EVEN FUNCTIE $\Rightarrow \displaystyle\int_{-\infty}^{+\infty} = 2 \int_0^{+\infty} = 2\pi i \sum Res$

de functie voldoet aan de voorwaarde en is singulier in $\pm 2i$ ($+2i \in$ BHV)

$$Res(f, 2i) = \lim_{z \to 2i} \frac{d}{dz}\left(\frac{1}{(z+2i)^2}\right) = \lim_{z \to 2i} \left(\frac{-2(z+2i)}{(z+2i)^4}\right) = -\frac{i}{32} \qquad \int_0^{+\infty} = \pi i \left(-\frac{i}{32}\right) = \frac{\pi}{32}$$

(#8) $\dfrac{1}{(x^2+1)^2(x^2+4)}$ IS EVEN $\Rightarrow \displaystyle\int_0^{+\infty} = \frac{1}{2} \int_{-\infty}^{+\infty} = \pi i \sum Res$

f^tie voldoet a/d voorwaarde en is singulier in $\pm i$, $\pm 2i$ ($+i$ en $+2i \in$ BHV)

$$\left.\begin{aligned} Res(f, i) &= \lim_{z \to i} \frac{d}{dz}\left(\frac{1}{(z+i)^2(z^2+4)}\right) = \lim_{z \to i}\left[-\frac{2(z+i)(z^2+4) + (z+i)^2 \cdot 2z}{(z+i)^4(z^2+4)^2}\right] = -\frac{i}{36} \\ Res(f, 2i) &= \lim_{z \to 2i} \frac{1}{(z^2+1)^2(z+2i)} = -\frac{i}{36} \end{aligned}\right\} \int_0^{+\infty} = \pi i \left(\frac{-i}{36} + \frac{i}{36}\right)$$

(#11) f^tie is even en voldoet aan de voorwaarde. singulier in $e^{i\left(\frac{\pi}{6} + k\frac{\pi}{3}\right)}$

$$\left.\begin{aligned} Res\left(e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i\right) &= -\frac{1}{6}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ Res\left(e^{i\frac{\pi}{2}} = i\right) &= -\frac{i}{6} \\ Res\left(e^{i\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) &= -\frac{1}{6}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \end{aligned}\right\} \int_0^{+\infty} = \pi i \left(\frac{-1}{6}\left(\cancel{\frac{\sqrt{3}}{2}} + \frac{1}{2}i + i - \cancel{\frac{\sqrt{3}}{2}} + \frac{1}{2}i\right)\right) = \frac{\pi}{3}$$

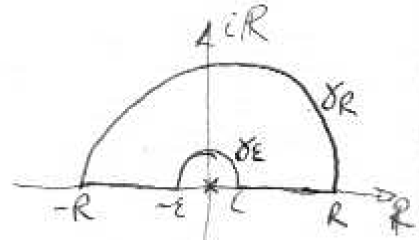
$e^{ix} = \overset{\cos}{\cancel{\sin}}\, x + i\, \overset{\sin}{\cancel{\cos}}\, x$

$$\int F(x) \sin x = \mathrm{Im}\left(\int F(x) e^{ix} \right)$$

$$\int F(x) \cos x = \mathrm{Re}\left(\int F(x) e^{ix} \right)$$

5.3 #1 $\displaystyle \int_0^{+\infty} \frac{\sin x \, dx}{x} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sin x}{x} dx = \mathrm{Im}\left(\frac{1}{2} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x} dx \right)$

$\displaystyle \int_{-\infty}^{+\infty} \frac{e^{ix}}{x} dx$ berekenen



$$\oint_\Gamma \frac{e^{ix}}{x} dx = 2\pi i \underset{0}{\underset{\shortparallel}{\mathrm{Res}}} = \int_{\gamma_R} + \int_{-R}^{-\varepsilon} + \int_{\gamma_\varepsilon} + \int_{\varepsilon}^{R}$$

F voldoet aan de voorwaarden $\Rightarrow \int_{\gamma_R}$ voor $R \to \infty$ $= 0$

$$\rightarrow \int_{-\infty}^{+\infty} \frac{e^{ix}}{x} = - \int_{\gamma_\varepsilon} = + \frac{1}{2} 2\pi i \,\mathrm{Res}(f, 0) = \pi i$$

γ_ε is met wijzers mee
residu is tegen de wijzers

$$\int_0^{+\infty} \frac{\sin x}{x} dx = \frac{1}{2} \mathrm{Im}(\pi i) = \frac{\pi}{2}$$

#5 $\displaystyle \int_0^{+\infty} \frac{x \sin x \, dx}{(x^2+1)(x^2+4)} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x \sin x}{(\ )(\ )} = I$

$\displaystyle I_0 = \oint_{\Gamma^+} \frac{x e^{ix}}{(\ )(\ )} = 2\pi i \,\mathrm{Re}$

$I = \frac{1}{2} \mathrm{Im}(I_0)$

Singulier in $\pm i$, $\pm 2i$ $\quad \displaystyle \oint_{\Gamma^+} = 2\pi i \left(\mathrm{Res}(f, i) + \mathrm{Res}(f, 2i) \right) \Rightarrow I = \mathrm{Im}\left(\frac{1}{2} 2\pi i \left(\frac{1}{6e} + \frac{1}{6e^2} \right) \right) = \frac{\pi}{6}\left(\frac{1}{e} + \frac{1}{e^2} \right)$

(6.1) (#1) $L(4e^{5t}+6t^3-3\sin 4t+2\cos 2t) = 4L(e^{5t})+6L(t^3)-3L(\sin 4t)+2L(\cos(2t))$

$= \frac{4}{p-5} + \frac{36}{p^4} - \frac{12}{p^2+16} + \frac{2p}{p^2-4}$

(#3) $L(t\,\mathrm{sh}\,2t)*$ $\quad L(t f(t)) = (-1)F'(p)$ $\quad * = (-1)\left(\frac{2}{p^2-4}\right)' = \frac{-4p}{(p^2-4)^2}$

(#5) $f(t) = \begin{cases} (t-1)^2 & t \geq 1 \\ 0 & t<1 \end{cases}$ $\Rightarrow L(f(t)) = e^{-p}L(t^2-2t+1) = e^{-p}\left(\frac{2}{p^3}-\frac{2}{p^2}+\frac{1}{p}\right)$

(#6) $f(t) = \int_0^t (u^2-u+e^{-u})du \cdot L(f(t)) = \frac{1}{p}\left(\frac{2}{p^3}-\frac{1}{p^2}+\frac{1}{p+1}\right)$

(#11) $f(t) = \frac{\sin 2t}{t}$ $\quad \lim_{t\to 0_+} \frac{\sin 2t}{t} \overset{H}{=} \lim_{t\to 0_+} 2\cos 2t = 2$

$\Rightarrow L\left(\frac{\sin 2t}{t}\right) = \int_p^\infty L(\sin 2t)dx = \int_p^\infty \frac{2}{x^2+4}dx = \int_p^\infty \frac{1}{2}\frac{dx}{\left(\frac{x}{2}\right)^2+1} = \left[\mathrm{Arctan}\frac{x}{2}\right]_p^\infty = \frac{\pi}{2} - \mathrm{Arc}$

(#16) $f(t) = \frac{\sin^2 t}{t}$ $\quad \lim_{t\to 0_+} \frac{\sin^2 t}{t} \overset{H}{=} \lim_{t\to 0_+} \frac{2\sin t\cos t}{1} = 0$

$\Rightarrow L\left(\frac{\sin^2 t}{t}\right) = \int_p^\infty L(\sin^2 t) = \int_p^\infty \left(\frac{1-\cos 2t}{2}\right) = \int_p^\infty \left[\frac{1}{2x} - \frac{1}{2}\left(\frac{x}{x^2+4}\right)\right]dx = \frac{1}{2}\left[\ln x\right]_p^\infty - \frac{1}{4}\left[\ln(x^2+4)\right]_p^\infty$

$= \left[\frac{1}{2}\ln x - \frac{1}{4}\ln(x^2+4)\right]_p^\infty = \left[\ln\frac{\sqrt{x}}{\sqrt[4]{x^2+4}}\right]_p^\infty = -\ln\frac{\sqrt{p}}{\sqrt[4]{p^2+4}} = \frac{1}{4}\ln\left(\frac{p^2+4}{p^2}\right)$

(6.2) (#1) periodische functie met periode 2

$\Rightarrow L(f(t)) = \frac{1}{1-e^{-2p}}\int_0^2 e^{-pt}t^2 dt = \frac{1}{1-e^{-2p}}\int_0^2 -\frac{1}{p}t^2 d e^{-pt} \overset{PI}{=} \frac{-1}{p(1-e^{-2p})}\left[\left[e^{-pt}t^2\right]_0^2 - \int_0^2 e^{-pt}\cdot 2t\right.$

$= \frac{-1}{p(1-e^{-2p})}\left[(e^{-2p}\cdot 4) - 2\int_0^2 -\frac{t}{p}de^{-pt}\right] \overset{PI}{=} \frac{-1}{p(1-e^{-2p})}\left[4e^{-2p} + \frac{2}{p}\left[\left[te^{-pt}\right]_0^2 - \int_0^2 e^{-pt}dt\right]\right]$

$= \frac{-1}{p(1-e^{-2p})}\left[4e^{-2p} + \frac{4e^{-2p}}{p} + \frac{1}{p^2}\left[e^{-pt}\right]_0^2\right] = \frac{-1}{(1-e^{-2p})}\left[\frac{4e^{-2p}}{p} + \frac{4e^{-2p}}{p^2} + \frac{2e^{-2p}}{p^3} - \frac{2}{p^3}\right]$

(#3) periodisch met periode π $\quad \Rightarrow L(f) = \frac{1}{1-e^{-\pi p}}\int_0^\pi e^{-pt}f(t)dt$

$= \frac{1}{1-e^{-\pi p}}\int_0^{\pi/2} e^{-pt}\cos^2 t\,dt = \frac{1}{1-e^{-\pi p}}\int_0^{\pi/2} e^{-pt}\left(\frac{\cos 2t+1}{2}\right)dt$

$= \frac{1}{1-e^{-\pi p}}\left[\frac{1}{2}\left(\int_0^{\pi/2} e^{-pt}\cos 2t\,dt + \int_0^{\pi/2} e^{-pt}dt\right)\right] = \frac{1}{1-e^{-\pi p}}\left[\frac{1}{2}\left(\frac{p(e^{-p\frac{\pi}{2}}+1)}{(p^2+4)} - \frac{(e^{-p\frac{\pi}{2}}-1)}{p}\right)\right]$

$\hookrightarrow$ 2* partiële integratie

6.3 #1 $\int_0^\infty t e^{-3t} \sin t \, dt = L(t \sin t)(3)$

$L(t\sin t) = (-1)\left(\frac{1}{p^2+1}\right)' = \frac{2p}{(p^2+1)^2} = \frac{6}{100} = \frac{3}{50}$

#2 $\int_0^\infty \frac{e^{-3t}\,\mathrm{sh}\, t}{t} dt = L\left(\frac{\mathrm{sh}\, t}{t}\right)(3)$ $\quad \lim_{t\to 0_+} \frac{\mathrm{sh}\, t}{t} \overset{H}{=} \lim_{t\to 0_+} \cosh t = \frac{\pi}{2}$

$L(p) = \int_0^\infty L(\mathrm{sh}\, x)\, dx = \int_0^\infty \frac{dx}{x^2-1}$

#4 $\int_0^\infty e^{-t} \cdot t^3 \sin t \, dt = L(t^3 \sin t)(1) = (-1)^3 \left(\frac{1}{p^2+1}\right)^{(3)}(1) = \dots = 0$

#5 $\int_0^\infty e^{-2t} \frac{1-\cos t}{t^2} dt = L\left(\frac{1-\cos t}{t^2}\right)(2)$ $\quad \lim_{t\to 0} \frac{1-\cos t}{t^2} = \infty \Rightarrow$ bestaat

$L\left(\frac{1-\cos t}{t^2}\right) = \int_p^\infty L\left(\frac{1-\cos t}{t}\right) dt = \int_p^\infty \int_t^\infty L(1-\cos x)\, dx\, dt$ $\quad \lim_{t\to 0} \frac{1-\cos t}{t} = 0 \Rightarrow$ bestaat

$\int_t^\infty L(1-\cos x)\, dx = \int_t^\infty \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \left[\ln x - \frac{1}{2}\ln(x^2+1)\right]_t^\infty = \left[\ln \frac{x}{\sqrt{x^2+1}}\right]_t^\infty = \frac{1}{2}\ln\left(\frac{t^2+1}{t^2}\right)$

$\int_p^\infty \frac{1}{2}\ln\left(\frac{t^2+1}{t^2}\right) dt = \frac{1}{2}\left[\left[t \ln\left(\frac{t^2+1}{t^2}\right)\right]_p^\infty - \int_p^\infty \frac{t^3}{t^2+1}\left(\frac{2t^3 - 2t(t^2+1)}{t^4}\right) dt\right]$

$= \frac{1}{2}\left[\left[\ln\left(\frac{t^2+1}{t^2}\right) t\right]_p^\infty + 2\int_p^\infty \frac{1}{t^2+1} dt\right] = \frac{p}{2}\ln\left(\frac{p^2}{p^2+1}\right) + \frac{\pi}{2} - \mathrm{Arctan}\, p$ $\quad p = 2 \Rightarrow \ln\left(\frac{4}{5}\right) + \frac{\pi}{2}$ -

#7 $\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t} dt = \int_0^\infty e^{-3t}\left(\frac{1-e^{-3t}}{t}\right) dt$ $\quad \lim_{t\to 0} \frac{1-e^{-3t}}{t} \overset{H}{=} \lim_{t\to 0} 3e^{-3t} = 3$

$\Rightarrow L\left(\frac{1-e^{-3t}}{t}\right) = \int_p^\infty L(1-e^{-3t})\, dt = \int_p^\infty \left(\frac{1}{t} - \frac{1}{t+3}\right) dt = \left[\ln t - \ln(t+3)\right]_p^\infty$

$= \ln \frac{p+3}{p}$ $\quad \cdot\ p = 0 \Rightarrow$ oplossing $= \ln 2$

WERKWIJZE : LAPLACETRANSFORMATIE NEMEN VAN BEIDE LEDEN

VERGELIJKING IN p OPLOSSEN

INVERSE LAPLACE NEMEN

HANDIG : $L\{y\} = Y(p)$ $L\{y'\} = pY - y(0)$ $L\{y''\} = p^2Y - py(0) - y'(0)$

7.1 #2 $y'' + y = 8\cos t \Rightarrow p^2Y - p\,y(0) - y'(0) + Y = 8\frac{p}{p^2+1}$

$\Rightarrow Y(p^2+1) = 8\frac{p}{p^2+1} + p - 1 \Rightarrow Y = \frac{8p}{(p^2+1)^2} + \frac{p-1}{(p^2+1)}$

$\frac{8p}{(p^2+1)^2} = -4\left(\frac{1}{p^2+1}\right)' \Rightarrow L^{-1}\left(\frac{8p}{(p^2+1)^2}\right) = 4t\sin t$ $\quad L^{-1}\left(\frac{p-1}{p^2+1}\right) = \cos t - \sin t$

$\Rightarrow$ opl $= 4t\sin t + \cos t - \sin t$

7.2 #1 $\begin{cases} pY + 1 + 2Y + Z = \frac{1}{p^2+1} \\ pZ - 1 - 4Y - 2Z = \frac{p}{p^2+1} \end{cases} \Rightarrow \begin{cases} Z = \frac{1}{p^2+1} - pY - 1 - 2Y \\ Y = \frac{-2}{p^2(p^2+1)} + \frac{1}{p^2} - \frac{1}{p} \end{cases}$

$L^{-1}(Y) = L^{-1}\left(\frac{-2}{p^2(p^2+1)} + \frac{1}{p^2}\right)^{-1} = L^{-1}\left(\frac{-2}{p^2} + \frac{2}{p^2+1} + \frac{1}{p^2}\right)^{-1} = L^{-1}\left(\frac{-1}{p^2}\right) + 2L^{-1}\left(\frac{1}{p^2+1}\right) = -t + 2\sin t -$

$L^{-1}(Z) = L^{-1}\left(\frac{1}{p^2+1}\right) - L^{-1}\left(\frac{-2}{p(p^2+1)} + \frac{1}{p} \cancel{-1} \cancel{+1}\right) - 2(-t + 2\sin t - 1) = \sin t - L^{-1}\left(\frac{-2}{p} + \frac{2p}{p^2+1} + \frac{1}{p}\right) - 2$

$= -3\sin t - 2\cos t + 3 + 2t$

7.4 #1 $L\left(\frac{\partial y}{\partial x}\right) \triangleq \int_0^\infty e^{-pt}\frac{\partial y}{\partial x}dt = \frac{\partial}{\partial x}\int_0^\infty e^{-pt}y\,dt = \frac{\partial Y}{\partial x}$

$L\left(\frac{\partial y}{\partial t}\right) \triangleq \int_0^\infty e^{-pt}\frac{\partial y}{\partial t}dt = L\{y'\} = pY - y(x,0)$

$\Rightarrow \frac{\partial y}{\partial x} = \frac{\partial y}{\partial t} + y$ wordt $\frac{\partial Y}{\partial x} = pY - y(x,0) + \frac{1}{p^2} = pY - 6e^{-3x} + \frac{1}{p^2}$

HOMOGEEN : $\frac{\partial Y}{\partial x} = pY \Rightarrow Y = C(p)e^{px}$

PARTICULIER :

7.4 #2 $\frac{\partial y}{\partial x} - \frac{\partial y}{\partial t} = 1 - e^{-t}$ wordt $\frac{\partial Y}{\partial x} - pY + x = \frac{1}{p} - \frac{1}{p+1}$

HOMOGEEN : $\frac{\partial Y}{\partial x} - pY = 0 \Rightarrow Y(x,p) = C(p)e^{px}$

PARTICULIER : $Y_{tot}(x,p) = C(p,x)e^{px}$ $\quad \frac{\partial C}{\partial x} = \left(-x + \frac{1}{p} - \frac{1}{p+1}\right)e^{-px}$

$\Rightarrow C(x,p) = \left(\frac{x}{p}e^{-px} + \frac{1}{p(p+1)}e^{-px} + B\right)$

$\Rightarrow Y_{tot} = \left(\frac{x}{p}e^{-px} + \frac{1}{p(p+1)}e^{-px} + B\right)e^{px} = \frac{x}{p} + \frac{1}{p(p+1)} + Be^{px}$

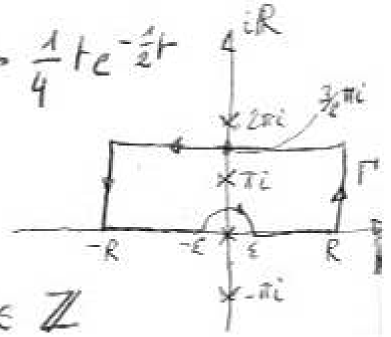
$\lim\limits_{p\to\infty} Y_{TOT} \underset{\forall x}{=} 0 \Rightarrow B = 0$

$L^{-1}(Y_{TOT}) = x + 1 - e^{-t}$

7.5 #4 $y(t) = t + 2\int_0^t y(u)\cos(t-u)du \Rightarrow Y = \frac{1}{p^2} + 2\left(Y\right)\left(\frac{p}{p^2+1}\right)$

$\Leftrightarrow Y\left(1 - \frac{2p}{p^2+1}\right) = \frac{1}{p^2} \Leftrightarrow Y = \frac{1}{p^2\left(1 - \frac{2p}{p^2+1}\right)}$

7.6 #2 $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \int_0^{\infty} \frac{f(t)\,dt}{1-e^{-t}}$ $*$ $\quad F(p) = \frac{1}{(2p+1)^2} \xrightarrow{L^{-1}} \frac{1}{4} t e^{-\frac{1}{2}t}$

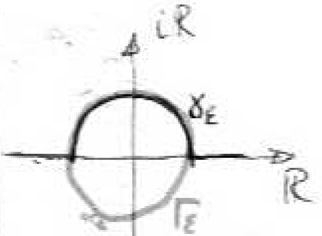


$\Rightarrow * = \frac{1}{4}\int_0^{\infty} \frac{t e^{-\frac{1}{2}t}}{1-e^{t}}\,dt = \frac{1}{4}\int_0^{\infty} \frac{t\,dt}{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}} = \cancel{1}\int_0^{\infty} \frac{z\,dz}{e^{z}-e^{-z}}$

singulier als $e^{z} = e^{-z} \Leftrightarrow e^{2z} = 1 \Leftrightarrow z = k\pi i$ met $k \in \mathbb{Z}$

$$\oint_{\Gamma} \frac{z\,dz}{e^{z}-e^{-z}} = 2\pi i \sum \mathrm{Res} = \int_{-R}^{-\varepsilon} + \int_{\gamma_\varepsilon} + \int_{\varepsilon}^{R} + \int_{\uparrow} + \int_{\leftarrow} + \int_{\downarrow}$$

het is een even functie $\Rightarrow \int_{\uparrow} + \int_{\downarrow} = 0$



$\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} = \int_{-R}^{R} \quad R\to\infty \quad \int_{-\infty}^{+\infty} = \frac{1}{2}\int_0^{+\infty}$ onbek. integraal

$\int_{\gamma_\varepsilon} = \frac{1}{2}\int_{\Gamma_\varepsilon} = \frac{1}{2} 2\pi i \,\mathrm{Res}(f, 0)$

$\int_{\leftarrow} \frac{z\,dz}{e^{z}-e^{-z}}$ (**) $z = x + \frac{3\pi}{2} i \Rightarrow ** = \int_{\leftarrow} \frac{(x+\frac{3}{2}\pi i)\,dx}{e^{(x+\frac{3}{2}\pi i)} - e^{(-x-\frac{3}{2}\pi i)}} = -i\int_{\leftarrow} \frac{(x+\frac{3}{2}\pi i)\,dx}{e^{x}+e^{-x}}$

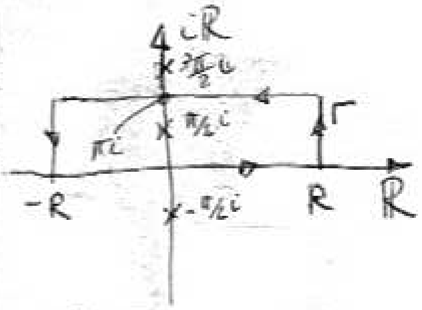
$= -i\int_{-R}^{R} \frac{x\,dx}{e^{x}-e^{-x}} + \int_{-R}^{R} \frac{\frac{3}{2}\pi\,dx}{e^{x}+e^{-x}} = \frac{3}{2}\pi\int_{-R}^{R} \frac{de^{x}}{e^{2x}+1} = \frac{3}{2}\pi\Big[\mathrm{Arctan}(e^{x})\Big]_{-R}^{R} \xrightarrow{R\to\infty} \frac{3\pi}{2}\left(\frac{\pi}{2}\right) = \frac{3\pi^2}{4}$

Oneven functie op symmetrisch interval $\Rightarrow \int = 0$

$\oint_{\Gamma} = 2\pi i\,\mathrm{Res}(f, \pi i) = \pi^2 = 2\cdot * + \frac{3\pi^2}{4} \Rightarrow * = \frac{\pi^2}{8}$

7.6 #5 $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}$ $*$ $\quad L^{-1}\left(\frac{1}{(2n-1)^3}\right) \longrightarrow \frac{1}{16} t^2 e^{\frac{1}{2}t} \qquad * = \frac{1}{16}\int_0^{+\infty} \frac{t^2\,dt}{e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}} = \frac{1}{8}\int \frac{z^2\,dz}{e^{z}+e^{-z}}$

SINGULIER ALS $e^{z} = -e^{-z} \Rightarrow z = i\frac{\pi}{2} + k\pi i$



$\oint_{\Gamma} = 2\pi i\,\mathrm{Res}(f, \frac{\pi}{2}i) = \int_{-R}^{R} + \int_{\uparrow} + \int_{\leftarrow} + \int_{\downarrow}$

even functie $\Rightarrow \int_{\uparrow} + \int_{\downarrow} = 0 \qquad \mathrm{Res}(f, \frac{\pi}{2}i) = \frac{\pi^2}{8} i$

$\int_{-R}^{R} \frac{z^2\,dz}{e^{z}+e^{-z}} \qquad z = x + \pi i \quad \Rightarrow \int_{-R}^{R} \frac{(x+\pi i)^2\,dx}{e^{(x+\pi i)} + e^{-(x+\pi i)}} \overset{**}{=} \int_{-R}^{R} \frac{x^2\,dx}{e^{x}+e^{-x}} - \int_{-R}^{R} \frac{\pi i x\,dx}{e^{x}+e^{-x}} + \int_{-R}^{R} \frac{\pi^2 \ldots}{e^{x}+e^{\ldots}}$

$\Rightarrow ** = \frac{1}{2}\int_{-R}^{R} \frac{\pi^2\,dx}{e^{x}+e^{-x}} = \frac{\pi^2}{2}\Big[\mathrm{Arctan}(e^{x})\Big]_{R}^{R} = \frac{\pi^3}{4}$

$\Rightarrow * = -\frac{\pi^3}{4} - \frac{\pi^3}{4} = -\frac{\pi^3}{2}$ NIET JUIST OPLOSSING IS $\frac{\pi^3}{32}$

8.1 #1 $\int_{-2}^{0} y + \sqrt{1+y'^2}\, dx$ met $A=(0,0)$ $B=(-2,0)$

onafhankelijk van x $\Rightarrow$ $y + \sqrt{1+y'^2} - y' \frac{2y'}{2\sqrt{1+y'^2}} = c$

$\Leftrightarrow$ $(y-c)\sqrt{1+y'^2} = -1$ $\Rightarrow$ $y' = \sqrt{\frac{1}{(y-c)^2} - 1}$

$X - A = \int \frac{y-c}{\sqrt{1-(y-c)^2}} dy$ $\quad u = 1-(y-c)^2$, $du = -2(y-c)dy$ $\Rightarrow$ $X - A = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{1-(y-c)^2}$

$\Rightarrow (x-A)^2 = 1-(y-c)^2$ cirkel met straal 1 en middelpt (A,c)

hij moet door $(0,0)$ en $(-2,0)$ gaan $\Rightarrow$ $A=-1$ $c=0$

8.1 #4 $I = \int_0^2 \frac{\sqrt{1+y'^2}}{x} dx$ onafhankelijk van y $\Rightarrow$ $\frac{\partial f}{\partial y'} = c$

$\frac{1}{x}\frac{y'}{\sqrt{1+y'^2}} = c$ $\Leftrightarrow$ $y' = \sqrt{\frac{x^2c^2}{1-x^2c^2}}$ $\Rightarrow$ $y - A = \int \sqrt{\frac{x^2c^2}{1-x^2c^2}} dx$

$\Rightarrow y - A = \int \frac{xc}{2c^2} \frac{1}{\sqrt{1-x^2c^2}} dx^2c^2 = \frac{1}{c}\sqrt{1-x^2c^2}$ $\quad (y-A)^2 = \frac{1}{c} - x^2$

BEGINVOORWAARDEN INVULLEN GEEFT $(y-2)^2 + x^2 = 4$

8.6 #1 $I = \int_{t_0}^{t_1} \left(\frac{m(y')^2}{2} - mgy \right) dt$

#2 $I = \int_{-1}^{1} \frac{\sqrt{dx^2 + dy^2}}{cy} = \int_{-1}^{1} \frac{\sqrt{1+y'^2}}{cy} dx$

#4 $I = \int_{x_0}^{x_1} 2\pi y \sqrt{1+y'^2}\, dx$

#5 $I = \int_0^{2\pi} e^{-y} \sqrt{1+y'^2}\, dx$

ONAFHANKELIJK VAN X $\rightarrow$ $f - y' \frac{\partial f}{\partial y'} = c$

$$e^{-y}\sqrt{1+y'^2} - y' \frac{y' e^{-y}}{\sqrt{1+y'^2}} = c$$

$$\Leftrightarrow 1 \cancel{+y'^2} \cancel{-y'^2} = ce^{y}\sqrt{1+y'^2} \Leftrightarrow \left(\frac{e^{-y}}{c}\right)^2 = 1 + y'^2$$

$$\Leftrightarrow y' = \sqrt{\frac{e^{-2y} - c^2}{c^2}}$$